A class of *N*-body problems with nearest- and next-to-nearest-neighbour interactions

2001 J. Phys. A: Math. Gen. 34 695

(http://iopscience.iop.org/0305-4470/34/4/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 695–724 www.iop.org/Journals/ja PII: S0305-4470(01)15046-8

A class of N-body problems with nearest- and next-to-nearest-neighbour interactions

Guy Auberson[1], **Sudhir R Jain**[2,4] **and Avinash Khare**[3]

[1] Physique Mathématique et Théorique, UMR 5825-CNRS, Université de Montpellier II, Montpellier, France
[2] Theoretical Physics Division, Bhabha Atomic Research Centre, Trombay, Mumbai 400 085, India
[3] Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, Orissa, India

E-mail: auberson@lpm.univ-montp2.fr, srjain@apsara.barc.ernet.in and khare@iopb.res.in

Received 26 June 2000, in final form 8 December 2000

Abstract
We obtain the exact ground state and part of the excitation spectrum in one dimension on a line and the exact ground state on a circle in the case where the N particles are interacting via nearest- and next-to-nearest-neighbour interactions. Furthermore, using the exact ground state, we establish a mapping between these N-body problems and the short-range Dyson models introduced recently to model intermediate spectral statistics. Using this mapping we compute the one- and two-point functions of a related many-body theory in the thermodynamic limit and show the absence of long-range order. However, quite remarkably, we prove the existence of an off-diagonal long-range order in the symmetrized version of the related many-body theory. Generalization of the models to other root systems is also considered. Besides, we also generalize the model on the full line to higher dimensions. Finally, we consider a model in two dimensions in which all the states exhibit novel correlations.

PACS numbers: 0530, 0375F, 0540

1. Introduction

In recent years, the Calogero–Sutherland- (CSM-) type N-body problems [1, 2] in one dimension have attracted considerable attention not only because they are exactly solvable [3] but also due to their relationship with (1 + 1)-dimensional conformal field theory, random matrix theory [4], etc. In particular, the connections between exactly solvable models [5] and random matrix theory [6] have been very fruitful. For example, by mapping these models to random matrices from an orthogonal, unitary or symplectic Gaussian ensemble, Sutherland [2] was able to obtain all static correlation functions of the corresponding many-body theory. The key point of this model is the pairwise long-range interaction among the N particles.

[4] Author to whom correspondence should be addressed.

One may add here that the family consisting of exactly solvable models, related to fully integrable systems, is quite small [3] and their importance lies in the fact that their small perturbations describe a wide range of physically interesting situations. Furthermore, recent developments [7] relating equilibrium statistical mechanics to random matrix theory owing to non-integrability of dynamical systems have made the pursuit of the unifying seemingly disparate ideas a very important theme. The results presented in this paper belong to the emerging intersection of several frontiers such as quantum chaos, random matrix theory, many-body theory and equilibrium statistical mechanics [8].

The universality in level correlations in linear (Gaussian) random matrix ensembles agrees very well with those in chaotic quantum systems [9] as also in many-body systems such as nuclei [6]. On the other hand, random matrix theory was connected to the world of exactly solvable models when the Brownian motion model was presented by Dyson [10], and later on, by the works on level dynamics [11]. However, there are dynamical systems which are neither chaotic nor integrable—the so-called pseudo-integrable systems [12]. It is known that the spectral statistics of such systems are 'non-universal with a universal trend' [13]. In particular, for Aharonov–Bohm billiards, the level spacing distribution is linear for small spacing and it falls off exponentially for large spacing [14]. Similar features are observed numerically for the Anderson model in three dimensions at the metal–insulator transition point [15]. To understand these statistical features, and in the context of random banded matrices, a new random matrix model (which has been called the short-range Dyson model in [16]) was introduced [17, 18], wherein the energy levels are treated as in the Coulomb gas model with the difference that only nearest neighbours interact. This new model explains features of intermediate statistics [16] in some polygonal billiards.

In view of all this it is worth enquiring whether one can construct an N-body problem which is exactly solvable and which is connected to the short-range Dyson model (SRDM)? If possible, then using this correspondence one can hope to calculate the correlation functions of the corresponding many-body theory and see whether the system exhibits long-range order and/or off-diagonal long-range order.

The purpose of this paper is to present two such models in one dimension, one on a line and the other on a circle. We obtain the exact ground state and part of the excitation spectrum on a line and the exact ground state on a circle in the case where the N particles are interacting via nearest- and next-to-nearest-neighbour interactions[5]. Furthermore, in both cases we show how the norm of the ground-state wavefunction is related to the joint probability density function of the eigenvalues of short-range Dyson models. Using this mapping, we obtain one- and two-point functions of a related many-body theory in the thermodynamic limit and prove the absence of long-range order in the system. However, quite remarkably, we prove the existence of an off-diagonal long-range order in the symmetrized version of the corresponding many-body theory[6].

We also extend this work in several different directions. For example, we consider an N-body problem with nearest- and next-to-nearest-neighbour interaction in an arbitrary number of dimensions D and show that the ground state and a part of the excitation spectrum can still be obtained analytically. We also obtain part of the bound state spectrum in one dimension (both on a full line and on a circle) by replacing the root system A_{N-1} by BC_N, D_N, etc. Besides, we also consider a model in two dimensions for which novel correlations are present in the ground as well as the excited states.

The plan of the paper is the following. In section 2 we consider an N-body problem on a

[5] A short account of this work has been given in [19].
[6] A short account of this work has been given in [20].

line characterized by the Hamiltonian (throughout this paper we shall use $\hbar = m = 1$)

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{i=1}^{N-1}\frac{1}{(x_i - x_{i+1})^2} - G\sum_{i=2}^{N-1}\frac{1}{(x_{i-1} - x_i)(x_i - x_{i+1})} + V\left(\sum_{i=1}^{N} x_i^2\right) \qquad (1)$$

with $G \geqslant 0$, while $g > -\frac{1}{4}$ to prevent the collapse that a more attractive inversely quadratic potential would cause. We show that the ground state and at least a part of the excitation spectrum can be obtained if

$$g = \beta(\beta - 1) \qquad G = \beta^2 \qquad V = \frac{\omega^2}{2}\sum_{i=1}^{N} x_i^2. \qquad (2)$$

Note that with the above restriction on G and g, $\beta \geqslant \frac{1}{2}$. Furthermore, we also point out the connection between the norm of the ground-state wavefunction and the joint probability distribution function for eigenvalues in SRDM. In section 3 we consider another N-body problem, but this time on a circle characterized by the Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\frac{\pi^2}{L^2}\sum_{i=1}^{N}\frac{1}{\sin^2[\frac{\pi}{L}(x_i - x_{i+1})]}$$
$$-G\frac{\pi^2}{L^2}\sum_{i=1}^{N}\cot\left[(x_{i-1} - x_i)\frac{\pi}{L}\right]\cot\left[(x_i - x_{i+1})\frac{\pi}{L}\right] \qquad (x_{N+1} = x_1) \qquad (3)$$

(where again $G \geqslant 0$ while $g > -\frac{1}{4}$) and obtain the exact ground state in the case where g and G are again as related by equation (2). Furthermore, we also point out the connection between the norm of the ground-state wavefunction and the joint probability distribution function for eigenvalues of the short-range circular Dyson model (SRCDM). Using this connection, in sections 4 and 5 we obtain several exact results for the corresponding many-body theory in the thermodynamic limit. In particular, in section 4 we calculate the two-particle correlation functions of a related many-body theory in the thermodynamic limit and prove the absence of long-range order in the system. In section 5 we consider the symmetrized version of the model considered in section 3 and show the existence of an off-diagonal long-range order in the bosonic system in the thermodynamic limit. In section 6 we consider the BC_N generalization of the model (1) and obtain the exact ground state of the system. In section 7 we consider the BC_N generalization of the model (3) and obtain the exact ground state of the system. In section 8 we consider a generalization of the model (1) to higher dimensions and obtain the ground state and a part of the excitation spectrum. In section 9 we consider a variant of the model (1) in two dimensions and obtain the ground state as well as a class of excited states, all of which have a novel correlation built into them. Finally, in section 10 we summarize the results obtained and point out several open questions.

2. N-body problem in one dimension on a line

Let us start from the Hamiltonian (1) and restrict our attention to the sector of configuration space corresponding to a definite ordering of the particles, say

$$x_i \geqslant x_{i+1} \qquad i = 1, 2, \ldots, N-1. \qquad (4)$$

On using the ansatz

$$\psi = \phi\prod_{i=1}^{N-1}(x_i - x_{i+1})^{\beta} \qquad (5)$$

in the corresponding Schrödinger equation $H\psi = E\psi$, it is easily shown that, provided g and G are related to β by equation (2), ϕ satisfies the equation

$$-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2\phi}{\partial x_i^2} - \beta\sum_{i=1}^{N-1}\frac{1}{(x_i - x_{i+1})}\left(\frac{\partial\phi}{\partial x_i} - \frac{\partial\phi}{\partial x_{i+1}}\right) + (V - E)\phi = 0. \tag{6}$$

Following Calogero we start from ϕ as given by equation (6) and assume that

$$\phi = P_k(x)\Phi(r) \tag{7}$$

where $r^2 = \sum_{i=1}^{N} x_i^2$. The function, Φ satisfies the equation

$$\Phi''(r) + [N + 2k - 1 + 2(N - 1)\beta]\frac{1}{r}\Phi'(r) + 2[E - V(r)]\Phi(r) = 0 \tag{8}$$

provided $P_k(x)$ is a homogeneous polynomial of degree k ($k = 0, 1, 2, \ldots$) in the particle coordinates and satisfies the generalized Laplace equation

$$\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\beta\sum_{i=1}^{N-1}\frac{1}{(x_i - x_{i+1})}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)\right]P_k(x) = 0. \tag{9}$$

We shall discuss few solutions of the Laplace equation (9) below.

Let us now specialize to the case of the oscillator potential, i.e. $V(r) = \frac{1}{2}\omega^2 r^2$. In this case, equation (8) is the well known radial equation for the oscillator problem in more than one dimension and its solution is

$$\Phi(r) = \exp(-\omega r^2/2)L_n^a(\omega r^2) \qquad n = 0, 1, 2, \ldots \tag{10}$$

where $L_n^a(x)$ is the associated Laguerre polynomial, while the energy eigenvalues are given by

$$E_n = \left[2n + k + \tfrac{1}{2}N + (N - 1)\beta\right]\omega = E_0 + (2n + k)\omega \tag{11}$$

with $a = E/\omega - 2n - 1$. A few comments are in order at this stage.

(a) For large N, the energy E is proportional to N so that

$$\lim_{N\to\infty}\frac{E}{N} = \left(\beta + \tfrac{1}{2}\right)\omega \tag{12}$$

i.e. the system has a good thermodynamic limit.

(b) The spectrum can be interpreted as being due to non-interacting bosons (or fermions) plus (n, k)-independent (but N-dependent) shift.

The ground-state eigenvalue and eigenfunction of the model is thus given by ($n = k = 0$)

$$E_0 = \left[(N - 1)\beta + \tfrac{1}{2}N\right]\omega \tag{13}$$

$$\psi_0 = \exp\left(-\frac{\omega}{2}\sum_{i=1}^{N} x_i^2\right)\prod_{i=1}^{N-1}(x_i - x_{i+1})^{\beta}. \tag{14}$$

A neat way of proving that we have indeed obtained the ground state can be given using the method of supersymmetric quantum mechanics [21]. To this end, we define the operators

$$Q_i = \frac{\mathrm{d}}{\mathrm{d}x_i} + \omega x_i + \beta\left[\frac{1}{(x_{i-1} - x_i)} - \frac{1}{(x_i - x_{i+1})}\right] \qquad (i = 2, 3, \ldots, N - 1)$$

$$Q_1 = \frac{\mathrm{d}}{\mathrm{d}x_1} + \omega x_1 - \beta\frac{1}{x_1 - x_2} \tag{15}$$

$$Q_N = \frac{\mathrm{d}}{\mathrm{d}x_N} + \omega x_N + \beta\frac{1}{x_{N-1} - x_N}$$

and their Hermitian conjugates Q_i^+. It is easy to see that the Qs annihilate the ground state as given by equation (14). Furthermore, the Hamiltonian (1) can be written in terms of these operators as

$$H - E_0 = \frac{1}{2}\sum_{i=1}^{N} Q_i^+ Q_i \tag{16}$$

where E_0 is as given by equation (13). Now since the operator on the right-hand side is non-negative and annihilates the ground-state wavefunction as given by equation (14), hence E_0 as given by equation (13) must be the ground-state energy of the system.

On rewriting ψ_0 in terms of a new variable

$$y_i \equiv \sqrt{\frac{\omega}{\beta}}x_i \tag{17}$$

one finds that the probability distribution for N particles is given by

$$\psi_0^2 = C\exp\left(-\beta\sum_{i=1}^{N} y_i^2\right)\prod_{i=1}^{N-1}(y_i - y_{i+1})^{2\beta} \tag{18}$$

where C is the normalization constant. We now observe that for $\beta = 1, 2, 4$, this ψ^2 can be identified with the joint probability density function for the eigenvalues of SRDM with Gaussian orthogonal, unitary or symplectic ensembles, respectively. We can therefore borrow the well known results for these ensembles [17, 18] and obtain exact results about a many-body theory defined in the limit, $N \to \infty$, $\omega \to 0$, $N\omega =$ finite, which defines the density of the system. For example, as $N \to \infty$, the one-point correlation function tends to a Gaussian for any β [17] and is given by

$$R_1(x) = \frac{N}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{19}$$

where $\sigma^2 = (\beta + 1)/\omega$. Here the k-point correlation function is defined as

$$R_k(x_1, \ldots, x_k) = \frac{N!}{(N-k)!}\int_{-\infty}^{\infty} \mathrm{d}x_{k+1}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_N\, P(x_1, x_2, \ldots, x_N). \tag{20}$$

In (20), the joint probability density function $P(x_1, x_2, \ldots, x_N)$ represents the joint probability of finding energy levels (or positions of particles) at x_i around $x_i + \mathrm{d}x_i$ ($i = 1, 2, \ldots, N$). From this basic definition, one can obtain the k-point correlation function and spacing distributions.

Other results concerning the many-body theory will be discussed in sections 4 and 5.

Finally, let us discuss the polynomial solutions to the Laplace equation (9). So far, we have been able to obtain solutions in the following cases:

(a) $k = 2$, $N \geqslant 2$;
(b) $k = 3$, $N \geqslant 3$;
(c) $k = 4$, $N \geqslant 4$;
(d) $k = 5$, $N \geqslant 5$;
(e) $k = 6$, $N \geqslant 6$.

Besides, we have also obtained solutions for $k = 4, 5, 6$ in the case $N = 3$, and for $k = 5, 6$ in the case $N = 4$. We find that for $k \geqslant 3$, the demand that there be no pole in $P_k(x)$ alone does not require $P_k(x)$ to be a completely symmetrical polynomial. However, for $k = 3, 4$ and $N = 3, 4$ it turns out that the solution to the Laplace equation (9) exists only if $P_k(x)$ is a completely symmetric polynomial. We suspect that this may be true in general. On assuming completely symmetric $P_k(x)$ we find that in all the above cases we have a one-parameter family of solutions. In particular, the various solutions are as follows (it is understood that the particle indices $i, j, k, \ldots$ are always unequal unless mentioned otherwise).

(a) $k = 2, N \geqslant 2$

$$P_k(x) = a\sum_{i=1}^{N} x_i^2 + b\sum_{i<j}^{N} x_i x_j \tag{21}$$

with β given by

$$\beta = \frac{aN}{(N-1)(b-2a)}. \tag{22}$$

(b) $k = 3, N \geqslant 3$

$$P_k(x) = a\sum_{i=1}^{N} x_i^3 + b\sum_{i,j=1}^{N} x_i^2 x_j + c\sum_{i<j<k}^{N} x_i x_j x_k \tag{23}$$

where $c = 3(b-a)$ and β is given by

$$\beta = \frac{3a + (N-1)b}{(N-1)(b-3a)}. \tag{24}$$

(c) $k = 4, N \geqslant 4$

$$P_k(x) = a\sum_{i=1}^{N} x_i^4 + b\sum_{i,j=1}^{N} x_i^3 x_j + c\sum_{i<j}^{N} x_i^2 x_j^2 + d\sum_{i,j<k}^{N} x_i^2 x_j x_k + e\sum_{i<j<k<l}^{N} x_i x_j x_k x_l \tag{25}$$

where

$$\begin{gathered} e = 6(c-2a) \qquad d = b + 2c - 4a \\ (N+4)b + 2(N-2)c - 4(N-2)a + 2(N-1)(2a+b-c)\beta = 0 \end{gathered} \tag{26}$$

and β is given by

$$\beta = \frac{6a + (N-1)c}{(N-1)(b-4a)}. \tag{27}$$

(d) $k = 5, N \geqslant 5$

$$\begin{aligned} P_k(x) = a\sum_{i=1}^{N} x_i^5 &+ b\sum_{i,j=1}^{N} x_i^4 x_j + c\sum_{i,j=1}^{N} x_i^3 x_j^2 + d\sum_{i,j<k}^{N} x_i^3 x_j x_k \\ &+ e\sum_{k,i<j}^{N} x_i^2 x_j^2 x_k + f\sum_{i,j<k<l}^{N} x_i^2 x_j x_k x_l + g\sum_{i<j<k<l<m}^{N} x_i x_j x_k x_l x_m \end{aligned} \tag{28}$$

where

$$\begin{gathered} e = 5c - 5a - 3b \qquad d = b + 2c - 5a \\ f = 12c - 15a - 9b \qquad g = 30(c-a-b) \\ (5N-7)c - 3(N-4)b - 5(N-2)a + (N-1)(5a+3b-2c)\beta = 0 \end{gathered} \tag{29}$$

and β is given by

$$\beta = \frac{10a + (N-1)c}{(N-1)(b-5a)}. \tag{30}$$

(e) $k = 6, N \geqslant 6$

$$\begin{aligned} P_k(x) = {} & a \sum_{i=1}^{N} x_i^6 + b \sum_{i,j=1}^{N} x_i^5 x_j + c \sum_{i,j=1}^{N} x_i^4 x_j^2 + d \sum_{i,j<k}^{N} x_i^4 x_j x_k + e \sum_{i<j}^{N} x_i^3 x_j^3 \\ & + f \sum_{ij,k=1}^{N} x_i^3 x_j^2 x_k + g \sum_{i,j<k<l}^{N} x_i^3 x_j x_k x_l + h \sum_{i<j<k}^{N} x_i^2 x_j^2 x_k^2 \\ & + p \sum_{i<j,<k<l}^{N} x_i^2 x_j^2 x_k x_l + q \sum_{i,j<k<l<m}^{N} x_i^2 x_j x_k x_l x_m \\ & + r \sum_{i<j<k<l<m<n}^{N} x_i x_j x_k x_l x_m x_n \end{aligned} \tag{31}$$

where

$$\begin{aligned} & 3e = 4b - 2c + 6a + f \qquad d = b + 2c - 6a \qquad g = 2f + 2c - 4b - 6a \\ & h = 2f + 9a - 4b - c \qquad p = 5f + 18a - 8b - 6c \\ & q = 6(2f + 9a - 4b - 3c) \qquad r = 30(f + 6a - 2b - 2c) \\ & (5N - 9)f - 2(4N - 15)b + 18(N - 5)a - 6(N - 5)c \\ & \qquad + (N - 1)(8b + 6c - 2f - 18a)\beta = 0 \\ & 14b - 2c + 6a + (N - 1)f + 2(N - 1)(3a + 2b - c)\beta = 0 \end{aligned} \tag{32}$$

and β is given by

$$\beta = \frac{15a + (N - 1)c}{(N - 1)(b - 6a)}. \tag{33}$$

It would be nice if one could find solutions for higher values of k and further check whether solutions exist (if at all) only if $P_k(x)$ is a completely symmetric polynomial. While we are unable to prove it, we suspect that, subject to the solutions of the Laplace equation for higher k, we have obtained the complete spectrum for this problem.

Finally, it is worth enquiring whether the bound state spectrum of the Hamiltonian (1) can also be obtained in the case where the oscillator potential is replaced by any other potential. It turns out that as in the Calogero case [22], in this case the answer to the question is also in the affirmative. In particular, if instead the N particles are interacting via the N-body potential as given by

$$V(x_1, x_2, \ldots, x_N) = -\alpha \sum_{i=1}^{N} \frac{1}{\sqrt{\sum_i x_i^2}} \tag{34}$$

then also (most likely the entire) discrete spectrum can be obtained. This is because, after using the ansatz (7), equation (8) is essentially the radial Schrödinger equation for an attractive Coulomb potential and it is well known that the only two problems which are analytically solvable for all partial waves are the Coulomb and the oscillator potentials. In particular, the solution of (8) is then given by (note $r^2 = \sum_{i=1}^{N} x_i^2$)

$$\Phi(r) = \exp(-\sqrt{2|E|}r) L_n^b(2\sqrt{2|E|}r) \tag{35}$$

and the corresponding energy eigenvalues are

$$E_{n,k} = -\frac{\alpha^2}{2\left[n+k+\frac{1}{2}N-1+(N-1)\beta\right]^2} \tag{36}$$

when $b = N + 2k - 3 + 2(N-1)\beta$. It may be noted that, whereas in the oscillator case the spectrum is linear in β, it is $(-E)^{-1/2}$ which is linear in β in the case of the Coulomb-like potential. Secondly, as in any oscillator (Coulomb) problem, the energy depends on n and k only through the combination $2n + k$ $(n + k)$.

Is there any underlying reason why one is able to obtain the discrete spectrum for the N-body problem with either the oscillator or the Coulomb-like potential (34)? Following [23] it is easily shown that in both the cases one can write down an underlying $su(1,1)$ algebra. Furthermore, since the many-body potential W in (1) is a homogeneous function of the coordinates of degree -2, i.e. it satisfies

$$\sum_{l=1}^{N} x_l \frac{\partial W}{\partial x_l} = -2W \tag{37}$$

hence, following the arguments of [23], one can also establish a simple algebraic relationship between the energy eigenstates of the N-body problem (1) with the Coulomb-like potential (34) and the harmonic oscillator potential.

It may be noted that the Hamiltonian (1) is not completely symmetric in the sense that, whereas all other particles have two neighbours, particles 1 and N have only one neighbour. Can one make it symmetric so that all particles will be treated on the same footing? One possible way is to add some extra terms in H. For example, consider

$$H_1 = H + H' \tag{38}$$

where H is as given by equation (1) while H' has the form

$$H' = \frac{g}{(x_N - x_1)^2} - G\left[\frac{1}{(x_N - x_1)(x_1 - x_2)} + \frac{1}{(x_{N-1} - x_N)(x_N - x_1)}\right]. \tag{39}$$

Clearly, by adding these extra terms, the problem has become cyclic invariant for any N, while for $N = 3$ it is identical to the Calogero problem and hence is, in fact, completely symmetric under the interchange of any two of the three particle coordinates. It may be noted that in the thermodynamic limit, these extra terms are irrelevant.

We can again start from the ansatz (5) (but with $N - 1$ replaced by N) in the Schrödinger equation $H_1\psi = E\psi$ and using equation (2) we find that ϕ again satisfies equation (6), but with $N - 1$ in the second term being replaced by N. On further using the substitution as given by equation (7) one finds that Φ satisfies equation (8), but with the coefficient of the 2β term being N instead of $N - 1$, while $P_k(x)$ is again a homogeneous polynomial of degree k $(k = 0, 1, 2, \ldots)$ in the particle coordinates, which now satisfies instead of equation (9)

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\beta \sum_{i=1}^{N} \frac{1}{(x_i - x_{i+1})}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)\right] P_k(x) = 0 \tag{40}$$

with $x_{N+1} = x_1$.

How do the solutions to the Laplace equations (9) and (40) compare? For $N = 3$, equation (40) is identical to that of Calogero and for this case Calogero has already obtained the solutions for any k. For $N > 3$ and for $k \geqslant 3$, the demand that there be no pole in $P_k(x)$ alone does not require $P_k(x)$ to be a completely symmetrical polynomial. However, for

$k = 3, 4$ and $N = 4$ it again turns out that the solution to Laplace equation (40) exists only if $P_k(x)$ is a completely symmetric polynomial. We suspect that this may be true in general. On assuming completely symmetric $P_k(x)$ we have been able to obtain a two-parameter family of solutions in the case $k = 3, 4, 5, 6$ and $N \geqslant k$ (note that for equation (9) we have obtained only a one-parameter family of solutions). As an illustration, the solution for $N \geqslant 4$ and $k = 4$ is given by (it is understood that the particle indices $i, j, k, \ldots$ are always unequal)

$$P_k(x) = a\sum_{i=1}^{N} x_i^4 + b\sum_{i,j=1}^{N} x_i^3 x_j + c\sum_{i<j}^{N} x_i^2 x_j^2 + d\sum_{i,j<k}^{N} x_i^2 x_j x_k + e\sum_{i<j<k<l}^{N} x_i x_j x_k x_l \tag{41}$$

where

$$e = 2(2a + 2d - 2b - c) \tag{42}$$

$$6a + (N-1)c + \beta[8a - 2b + (2c - d)(N-2)] = 0 \tag{43}$$

$$6b + (N-2)d + 2\beta[2(N-1)b - 2(N-4)a + (N-4)c - 2(N-1)d] = 0. \tag{44}$$

The solution to the new Φ equation can be easily written down in the case $V(r) = \frac{1}{2}\omega^2 r^2$ or if it is given by equation (34). For example, it is easily checked that in the former case the solution is again given by equation (11) but the energy eigenvalues are now given by

$$E_{n,k} = [2n + k + \tfrac{1}{2}N + N\beta]\omega = E_0 + (2n + k)\omega. \tag{45}$$

Similarly, in the later case, the solution is as given by equation (36) except that in the term containing β, $N - 1$ must be replaced by N.

Apart from these two potentials, where we have obtained the entire bound state spectrum, there are several other potentials which are quasi-exactly solvable. For example, for the potential

$$V\left(\sum x_i^2\right) = A\sum_{i=1}^{N} x_i^2 - B\left(\sum_{i=1}^{N} x_i^2\right)^2 + C\left(\sum_{i=1}^{N} x_i^2\right)^3 \tag{46}$$

it is easily shown that the ground-state energy and eigenfunctions are

$$E = -\frac{B}{4\sqrt{C}}[N + 2(N-1)\beta] \tag{47}$$

$$\psi_0 = \exp\left[-\frac{\sqrt{C}}{4}\left(\sum_{i=1}^{N} x_i^2\right)^2 + \frac{B}{4\sqrt{C}}\sum_{i=1}^{N} x_i^2\right]\prod_{i=1}^{N-1}(x_i - x_{i+1})^\beta \tag{48}$$

provided A, B, C are related by

$$A = \frac{B^2}{4C} - [N + 2 + 2(N-1)\beta]\sqrt{C}. \tag{49}$$

It is worth enquiring whether the probability distribution for N particles corresponding to (48) can be mapped to some matrix model. In this context let us point out that the corresponding (long-ranged) Calogero problem was, in fact, mapped to the matrix model corresponding to branched polymers [24]. So far as we are aware of, the answer to this question is not known in our case.

3. N-body problem in one dimension with periodic boundary condition

Soon after the seminal papers of Calogero [1] and Sutherland [2] where they considered an N-body problem on a full line, Sutherland [25] also considered an N-body problem with long-ranged interaction and with a periodic boundary condition (PBC). He obtained the exact ground-state energy and showed that the corresponding N-particle probability density function is related to the random matrix in a circular ensemble [25]. Using the known results for random matrix theory [6], he was able to obtain the static correlation functions of the corresponding many-body theory. It is then natural to enquire whether one can also obtain the exact ground state of an N-body problem with nearest- and next-to-nearest-neighbour interaction with a periodic boundary condition. Furthermore, one would like to enquire whether the corresponding N-particle probability density can be mapped to some known matrix model. The hope is that in this case one may be able to obtain the correlation functions of a related many-body theory in the thermodynamic limit. We now show that the answer to the question is in the affirmative.

Let us start from the Hamiltonian (3). We wish to find the ground state of the system subject to the periodic boundary condition

$$\psi(x_1, \ldots, x_i + L, \ldots, x_N) = \psi(x_1, \ldots, x_i, \ldots, x_N). \tag{50}$$

For this, we start with a trial wavefunction of the form

$$\Psi_0 = \prod_{i=1}^{N} \sin^\beta \left[\frac{\pi}{L}(x_i - x_{i+1})\right] \qquad (x_{N+1} = x_1). \tag{51}$$

In this section, we restrict the coordinates x_i to the sector $L \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_N \geqslant 0$, so that equation (51) makes sense even for non-integer β. The extension to the full configuration space will be made in section 5. On substituting equation (51) in the Schrödinger equation for the Hamiltonian (3), we find that it is indeed a solution provided g and G are again related to β by equation (2). The corresponding ground-state energy turns out to be

$$E_0 = \frac{N\beta^2\pi^2}{L^2}. \tag{52}$$

The fact that this is indeed the ground-state energy can be neatly proved by using the operators [33]

$$Q_i = \frac{\mathrm{d}}{\mathrm{d}x_i} + \beta\frac{\pi}{L}\left[\cot(x_{i-1} - x_i) - \cot(x_i - x_{i+1})\right] \tag{53}$$

and their Hermitian conjugates Q_i^+. It is easy to see that the Qs annihilate the ground state as given by equation (51). The Hamiltonian (3) can be rewritten in terms of these operators as

$$H - E_0 = \tfrac{1}{2}\sum_i Q_i^+ Q_i \tag{54}$$

where E_0 is as given by equation (52). Hence E_0 must be the ground-state energy of the system.

Thus unlike the Calogero–Sutherland-type models, our models (both of section 2 and here) have a good thermodynamic limit, i.e. the ground-state energy per particle ($= E_0/N$) is finite as $N \to \infty$.

Having obtained the exact ground state, it is natural to enquire whether the corresponding N-particle probability density can be mapped to the joint probability distribution of some SRCDM so that we can obtain some exact results for the corresponding many-body theory.

It turns out that indeed the square of the ground-state wavefunction is related to the joint probability distribution function for the SRCDM from where we conclude that the density is a constant if $0 \leqslant x \leqslant N/L$, and zero outside. Other exact results for the many-body theory will be discussed in the next two sections.

4. Some exact results for the many-body problem

The square of the ground-state wavefunction of the many-body problem introduced in section 2 (section 3) can be identified with the joint probability distribution function of eigenvalues of the SRDM (SRCDM). Using SRCDM, Pandey [17] and Bogomolny *et al* [18] have shown that for any β, the two-point correlation function as defined by equation (20) has the form

$$R_2^{(\beta)}(s) = \sum_{n=1}^{\infty} P^{(\beta)}(n, s) \tag{55}$$

where s is the separation of two levels (or distance between two particles in the many-body theory considered here) and

$$P^{(\beta)}(n, s) = \frac{(\beta+1)^{n(\beta+1)}}{\Gamma[n(\beta+1)]} s^{n(\beta+1)-1} \mathrm{e}^{-(\beta+1)s}. \tag{56}$$

From this expression it is not very easy to compute $R_2(s)$ for arbitrary β. However, it is easy to obtain the Laplace transform of $R_2(s)$ for any β. In particular, if

$$g_2(t) = \int_0^{\infty} R_2(s)\, \mathrm{e}^{-ts}\, \mathrm{d}s \tag{57}$$

then

$$g_2(t) = \sum_{n=1}^{\infty} g(n, t) \tag{58}$$

where $g(n, t)$ is the Laplace transform of $P(n, s)$, i.e.

$$g(n, t) = \int_0^{\infty} P(n, s)\, \mathrm{e}^{-ts}\, \mathrm{d}s. \tag{59}$$

On using $P^{(\beta)}(n, s)$ as given by equation (56) in equation (59) it is easily shown that

$$g^{(\beta)}(n, t) = \left(\frac{\beta+1}{t+\beta+1}\right)^{(\beta+1)n}. \tag{60}$$

Hence

$$g_2^{(\beta)}(t) = \sum_{n=1}^{\infty} g^{(\beta)}(n, t) = \frac{1}{(\frac{t+\beta+1}{\beta+1})^{\beta+1} - 1} \tag{61}$$

from which one has to compute $R_2^{(\beta)}(s)$ by the Laplace inversion.

For integer β, it is possible to perform the Laplace inversion by making use of the fact that

$$\frac{1}{x^n - 1} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\mathrm{e}^{2\mathrm{i}k\pi/n}}{x - \mathrm{e}^{2\mathrm{i}k\pi/n}} \tag{62}$$

yielding

$$R_2^{(\beta)}(s) = \sum_{k=0}^{\beta} \Omega^k \mathrm{e}^{(\beta+1)s(\Omega^k - 1)} \tag{63}$$

where

$$\Omega = e^{2\pi i/(\beta+1)}. \tag{64}$$

For $\beta = 1$, which corresponds to the orthogonal ensemble, the result is already known [17, 18]: $R_2^{(1)}(s) = 1 - e^{-4s}$.

It is interesting to mention that $R_2^{(1)}(s)$ agrees very well with some of the pseudo-integrable billiards (e.g. the $\frac{1}{3}\pi$-rhombus billiard). It is important here to note that for rhombus billiards [16], the Hamiltonian matrix has elements which fall in their magnitude away from the principal diagonal. Thus, beyond a certain bandwidth, the elements are insignificant and the matrix is effectively banded. Immediately then, the results of banded matrices become applicable. Although there seems to be good agreement of the results from this random matrix theory as shown in [16, 18], in [16] it is also shown that there are other polygonal billiards for which $R_2^{(1)}(s)$ is not an appropriate correlator. It is possible that for different bandwidths, and, by an inclusion of interactions beyond nearest neighbours in the short-range Dyson model, a family of random matrices result. This may, eventually, explain the entire family of systems exhibiting intermediate spectral statistics.

Coming back to the two-point correlation function, depending on whether β is an odd or an even integer, $R_2(s)$, as given by equation (63), can be written in a closed form which shows that $R_2(s)$ is indeed real and further, it clearly exhibits oscillations for large s. In particular, it is easily shown that

$$\begin{aligned} R_2(\beta = 2p+1, s) = 1 - e^{-2(2p+2)s} + 2e^{-(2p+2)s} \sum_{m=1}^{p} \exp\left[(2p+2)s \cos\left(\frac{m\pi}{p+1}\right)\right] \\ \times \cos\left[\frac{m\pi}{p+1} + (2p+2)s \sin\left(\frac{m\pi}{p+1}\right)\right] \end{aligned} \tag{65}$$

$$\begin{aligned} R_2(\beta = 2p, s) = 1 + 2e^{-(2p+1)s} \sum_{m=1}^{p} \exp\left[(2p+1)s \cos\left(\frac{2m\pi}{2p+1}\right)\right] \\ \times \cos\left[\frac{2m\pi}{2p+1} + (2p+1)s \sin\left(\frac{2m\pi}{2p+1}\right)\right]. \end{aligned} \tag{66}$$

For illustration, we give below explicit expressions for $\beta = 2, 3, 4$:

$$\begin{aligned} R_2^{(2)}(s) &= 1 - 2e^{-9s/2} \cos\left(\frac{3\sqrt{3}s}{2} - \frac{\pi}{3}\right) \\ R_2^{(3)}(s) &= 1 - e^{-8s} - 2e^{-4s} \sin(4s) \\ R_2^{(4)}(s) &= 1 + 2e^{5s(-1+\cos(2\pi/5))} \cos\left[\frac{2\pi}{5} + 5s \sin\left(\frac{2\pi}{5}\right)\right] \\ &\quad + 2e^{5s(-1+\cos(4\pi/5))} \cos\left[\frac{4\pi}{5} + 5s \sin\left(\frac{4\pi}{5}\right)\right]. \end{aligned} \tag{67}$$

In figure 1, we have plotted $R_2^{(\beta)}(s)$ as a function of s for $\beta = 1, 2, 3, 4$. These results show that, for integer β, there is no long-range order in the corresponding many-body theory since $R_2(s)$ approaches 1 exponentially fast. The correlations do not grow beyond a certain scale, making phase separation impossible.

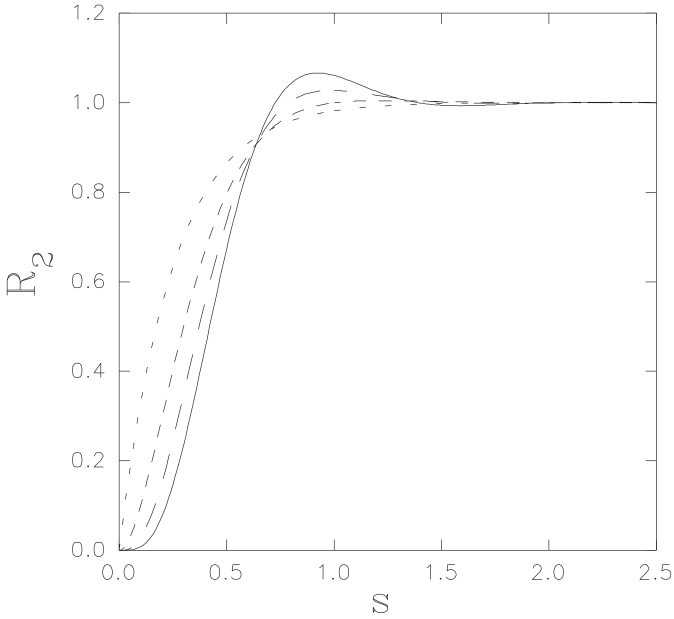

Figure 1. The two-point correlation function for four integer values of β (from left to right values increase from 1 to 4) clearly shows an absence of long-range order.

Similarly, if β is half-integral, i.e. $\beta = (2n+1)/2$ then it is easily shown that

$$R_2^{((2n+1)/2)}(s) = \frac{1}{2}\sum_{k=0}^{2n} \Omega^{2k} \mathrm{e}^{-\frac{1}{2}(2n+1)s(1-\Omega^{2k})}\left[1 + \mathrm{erf}\left(\sqrt{\frac{(2n+1)s}{2}}\Omega^k\right)\right] \qquad (68)$$

where Ω is as given by equation (64).

For arbitrary β, however, we are unable to perform the Laplace inversion and hence we do not have a closed expression for $R_2(s)$. However, one can calculate it numerically by using equations (55) and (56). In figure 2, we have plotted $R_2^{(\beta)}(s)$ as a function of s for $\beta = 1, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 2, \frac{7}{3}, \frac{5}{2}$. From this figure it is clear that even for fractional β, there is no long-range order.

5. Off-diagonal long-range order

So far, nothing has been specified regarding the statistical character of the particles involved in the N-body problem of section 3. We now do that by first symmetrizing the Hamiltonian, that is by rewriting it as

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{P\varepsilon S_N}\Theta(x_{P(1)} - x_{P(2)})\ldots\Theta(x_{P(N-1)} - x_{P(N)})W(x_{P(1)},\ldots,x_{P(N)}) \qquad (69)$$

where Θ is the step function and $W(x_1,\ldots,x_N)$ is the N-body potential of equation (3). Next, relying on the solution given in equation (51), we introduce the (not normalized) wavefunction

$$\psi_N(x_1,\ldots,x_N) = \varepsilon_P\phi_N(x_{P(1)},\ldots,x_{P(N)}) \qquad (70)$$

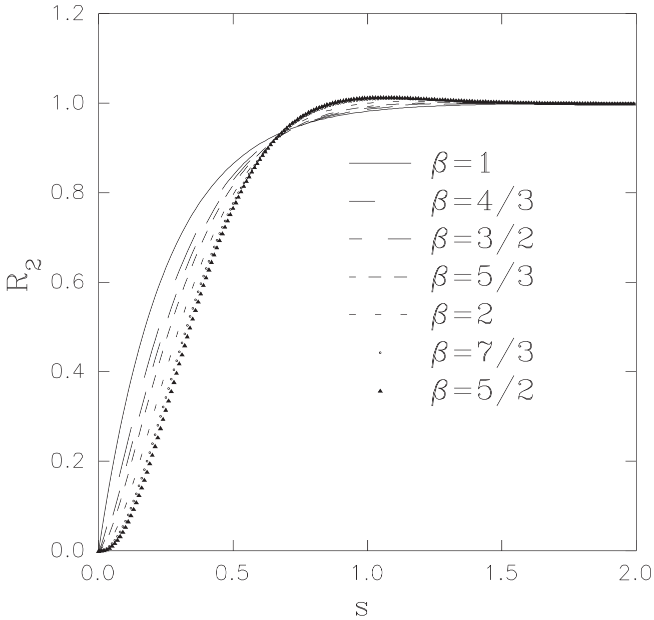

Figure 2. The two-point correlation function for some fractional values of β plotted along with β equal to 1 and 2. Values increase from left to right: 1, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$, 2, $\frac{7}{3}$ and $\frac{5}{2}$. Thus, even for fractional values, there is no long-range order.

where P is the permutation in S_N such that $1 > x_{P(1)} > x_{P(2)} > \cdots > x_{P(N)} > 0, \varepsilon_P = 1 (\varepsilon_P = \mathrm{sign}\,(P))$ in the N-boson (N-fermion) case and

$$\phi_N(x_1, \ldots, x_N) = \prod_{n=1}^{N} |\sin\pi(x_n - x_{n+1})|^\beta \qquad (x_{N+1} = x_1) \tag{71}$$

(we have set the scale factor L equal to 1). Primitively, the function (70) is defined on the hypercube $[0, 1]^N$. The following properties of ψ_N are easily verified, provided that $\beta \geqslant 2$

(a) In the bosonic case, ψ_N can be continued to a multi-periodic function over the whole space $\mathcal{R}^N$ (or equivalently on the torus T^N):

$$\psi_N(x_1, \ldots, x_i + 1, \ldots, x_N) = \psi_N(x_1, \ldots, x_i, \ldots, x_N) \qquad (i = 1, \ldots, N) \tag{72}$$

which belongs to C^2 (i.e. is twice continuously differentiable). Owing to this property and the results of section 3, ψ_N then obeys the Schrödinger equation (with Hamiltonian (3) and energy as given by equation (52)) not only in the sector $x_1 > x_2 > \cdots > x_N$ but everywhere. Thus, ψ_N describes the ground-state wavefunction of the N-boson system. Moreover, it is translation invariant (on $\mathcal{R}^N$):

$$\psi_N(x_1 + a, x_2 + a, \ldots, x_N + a) = \psi_N(x_1, x_2, \ldots, x_N) \qquad \forall a \in \mathcal{R}. \tag{73}$$

(b) In the fermionic case, the continuation by periodicity is possible only for *odd N*, in which case equation (72) still holds with $\psi_N \in C^2$. For even N in contrast, enforcing the periodicity (72) leads to a discontinuous function ψ_N, so that the Schrödinger equation is no longer satisfied on the configuration space T^N.
Therefore, in the following we shall implicitly restrict ourselves to odd values of N when treating fermions. The translation invariance (73) then remains valid.

We are interested in the one-particle reduced density matrix, given by

$$\rho_N(x-x') = \frac{N}{C_N}\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{N-1}\, \psi_N(x_1,\ldots,x_{N-1},x)\psi_N(x_1,\ldots,x_{N-1},x') \tag{74}$$

where C_N denotes the squared norm of the wavefunction:

$$C_N = \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_N\, |\psi_N(x_1,\ldots,x_N)|^2. \tag{75}$$

That the right-hand side of equation (74) defines a (periodic) function of $(x-x')$ is an easy consequence of equations (72) and (73). The normalization of ρ_N is such that $\rho_N(0) = N$, the particle density. Furthermore, the function $\rho_N(\xi)$ is manifestly of positive type on the $U(1)$ group, which implies that its Fourier coefficients,

$$\rho_N^{(n)} = \int_0^1 \mathrm{d}\xi\, \mathrm{e}^{-2\mathrm{i}\pi n\xi}\rho_N(\xi) \qquad (n = 0, \pm1, \pm2, \ldots) \tag{76}$$

are non-negative (Bochner's theorem). In fact, this appears directly if one writes their explicit expression

$$\rho_N^{(n)} = \frac{N}{C_N}\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{N-1}\, \psi_N(x_1,\ldots,x_{N-1},0)\int_0^1 \mathrm{d}x\, \mathrm{e}^{2\mathrm{i}\pi nx}\psi_N(x_1,\ldots,x_{N-1},x) \tag{77}$$

in the form (obtained by using the periodicity property):

$$\rho_N^{(n)} = \frac{N}{C_N}\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{N-1} \left|\int_0^1 \mathrm{d}x\, \mathrm{e}^{2\mathrm{i}\pi nx}\psi_N(x_1,\ldots,x_{N-1},x)\right|^2. \tag{78}$$

In the *bosonic* case, since the function ρ_N is not only of positive type but also *positive* (such as ψ_N), equation (76) shows us that

$$\rho_N^{(0)} \geqslant \rho_N^{(n)} \qquad (n = \pm1, \pm2, \ldots). \tag{79}$$

In the fermionic case, equation (79) is not necessarily true (because ψ_N changes sign on T^N) and it is not an easy matter to determine the largest Fourier coefficient. Note that the coefficients $\rho_N^{(n)}$, which physically represent the expectation values of the number of particles having momentum $k_n = 2\pi n$ in the ground state, are nothing but the eigenvalues of the one-particle reduced density matrix (diagonal in the k_n representation). According to the Onsager–Penrose criterion [26], no condensation can occur in the system (at least for Bose particles) if the largest of these eigenvalues is not an extensive quantity in the thermodynamic limit, that is, if

$$\lim_{N\to\infty}\frac{\rho_N^{(0)}}{N} = 0. \tag{80}$$

For Fermi particles, this criterion is not sufficient, and one has to also look at the largest eigenvalue of the two-particle reduced density matrix [28]. Since we are presently unable to determine the largest eigenvalue of ρ_N itself in the fermionic case, we shall not discuss the latter here extensively. Nevertheless, we shall look for the large-N behaviour of $\rho_N^{(0)}$ for bosons and fermions at a time, as this does not require much extra work and can give some indications in the fermionic case too. Let us write

$$\frac{\rho_N^{(0)}}{N} = \frac{A_N}{C_N} \tag{81}$$

where C_N is given by equation (75) and

$$A_N = \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{N-1}\, \psi_N(x_1, \ldots, x_{N-1}, 0) \int_0^1 \mathrm{d}x\, \psi_N\,(x_1, \ldots, x_{N-1}, x) \tag{82}$$

(the expression (77) of $\rho_N^{(0)}$ is more convenient than (78) for our purpose). Because of the special form (70) and (71) of the wavefunction, the computation of the squared norm C_N is already not a trivial task, in sharp contrast to the case of N free, impenetrable particles. Consequently, the (mainly algebraic) method introduced long ago by Lenard [27] to deal with the latter case does not apply here, and we have to resort to another device. For conciseness, we introduce the notation

$$S(x_n - x_{n-1}) {=} |\sin\pi(x_n - x_{n+1})|^\beta \tag{83}$$

and define

$$S_2(\triangle) = \int_0^\triangle \mathrm{d}x\, S(x)S(\triangle - x) \qquad (0 \leqslant \triangle \leqslant 1). \tag{84}$$

Our starting point will be the following representations of C_N and A_N:

$$C_N = (N-1)!\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x} \tilde{F}(x)^N \tag{85}$$

$$A_N = (N-1)!\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x} \tilde{F}(x)^{N-3}\big[\tilde{F}(x)\tilde{G}(x) + \eta_N \tilde{H}(x)^2\big] \tag{86}$$

where

$$\begin{aligned} \tilde{F}(x) &= \int_0^1 \mathrm{d}\triangle\, \mathrm{e}^{\mathrm{i}\triangle x} S(\triangle)^2 \\ \tilde{G}(x) &= \int_0^1 \mathrm{d}\triangle\, \mathrm{e}^{\mathrm{i}\triangle x} S_2(\triangle)^2 \\ \tilde{H}(x) &= \int_0^1 \mathrm{d}\triangle\, \mathrm{e}^{\mathrm{i}\triangle x} S(\triangle) S_2(\triangle)^2 \end{aligned} \tag{87}$$

and

$$\eta_N = \begin{cases} (N-2) & \text{for bosons} \\ -1 & \text{for fermions.} \end{cases} \tag{88}$$

The representations (85)–(88) follow from the convolution structure of the expressions (75) and (82) of C_N and A_N, when written in terms of appropriate variables. Their proof is given in the appendix. Our aim is to extract from them the large-N behaviour of C_N and A_N. Their form is especially suited for that purpose, because the integrands in equations (85) and (86) are entire functions, as polynomial combinations of Fourier transforms of functions with compact support (equation (87)). Indeed, we are then allowed to, first, shift the integration path and then apply the residue theorem to meromorphic pieces of the integrands. However, it turns out that the calculations needed for arbitrary (integer) values of β are quite cumbersome. So, in order to keep the argument clear enough, we shall content ourselves to presenting these calculations below in the simplest case, namely $\beta = 1$ (recall that, strictly speaking, this value is not allowed), it being understood that similar results are obtained for all integers $\beta \geqslant 2$.

For $\beta = 1$, $S(\triangle) = \sin \pi \triangle$, and equation (87) gives, after reductions:

$$\begin{aligned}
\tilde{F}(x) &= \frac{2\pi^2}{\mathrm{i}} \frac{1 - \mathrm{e}^{\mathrm{i}x}}{x(x^2 - 4\pi^2)} \\
\tilde{G}(x) &= \frac{4\pi^4}{\mathrm{i}} \frac{5x^2 - 4\pi^2}{x^3(x^2 - 4\pi^2)^3} + \mathrm{e}^{\mathrm{i}x} R^{(-1)}(x) \\
\tilde{H}(x) &= -\frac{4\pi^3}{\mathrm{i}} \frac{1}{x(x^2 - 4\pi^2)^2} + \mathrm{e}^{\mathrm{i}x} R^{(-2)}(x)
\end{aligned} \tag{89}$$

where $R^{(n)}(x)$ is a generic notation for rational functions behaving like x^n when $x \to \infty$, and the precise form of which will eventually be of no importance. This produces, for the functions to be integrated in equations (85) and (86),

$$\tilde{F}(x)^N = \left(\frac{2\pi^2}{\mathrm{i}}\right)^N \left[\frac{1}{[x(x^2 - 4\pi^2)]^N} + \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nx} R_n^{(-3N)}(x)\right] \tag{90}$$

$$\begin{aligned}
&\tilde{F}(x)^{N-3}[\tilde{F}(x)\tilde{G}(x) + \eta_N \tilde{H}(x)^2] \\
&\qquad = \mathrm{i}\left(\frac{2\pi^2}{\mathrm{i}}\right)^N \left\{\frac{(5 + 2\eta_N)x^2 - 4\pi^2}{[x(x^2 - 4\pi^2)]^{N+1}} + \sum_{n=1}^{N+1} \mathrm{e}^{\mathrm{i}nx} R_n^{(-3N-1)}(x)\right\}.
\end{aligned} \tag{91}$$

Let us stress again that these functions, when analytically continued, are holomorphic over the whole complex plane (the poles appearing in the first term are exactly cancelled by the remaining ones).

We consider first C_N, now given by

$$C_N = (N-1)!\left(\frac{2\pi^2}{\mathrm{i}}\right)^N \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}x} \left\{\frac{1}{[x(x^2 - 4\pi^2)]^N} + \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nx} R_n^{(-3N)}(x)\right\}. \tag{92}$$

Since the function within the curly brackets is an entire one, we can shift the integration path to $I \equiv \{z = x + \mathrm{i}a | x \varepsilon \mathcal{R}\}$. Let us choose $a > 0$. Then, by Cauchy's theorem

$$\int_I \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}z} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nz} R_n^{(-3N)}(z) = 0. \tag{93}$$

Indeed, the integrand is holomorphic above I and is bounded there by constant $|z|^{-3N}$, which allows us to close the integration path at infinity in the *upper* complex plane. We end up with

$$C_N = (N-1)!\left(\frac{2\pi^2}{\mathrm{i}}\right)^N \frac{1}{2\pi} \int_I \mathrm{d}z\, \frac{\mathrm{e}^{-\mathrm{i}z}}{z^N(z^2 - 4\pi^2)^N}. \tag{94}$$

Similarly, we are allowed to close the integration path at infinity in equation (94), but this time in the *lower* complex plane. The integrand now has poles at $z = 0, \pm 2\pi$, and applying the residue theorem leads to explicit expressions for C_N. Unfortunately, these expressions turn out to appear as (finite) sums with alternating signs, the terms of which become very close to each other for large N. They are therefore useless for determining the asymptotic behaviour of C_N, and we have to proceed differently. Let us write

$$\begin{aligned}
\int_I \mathrm{d}z\, \frac{\mathrm{e}^{-\mathrm{i}z}}{z^N(z^2 - 4\pi^2)^N} &= \frac{1}{(N-1)!} \frac{\mathrm{d}^{N-1}}{\mathrm{d}\alpha^{N-1}}\bigg|_{\alpha=4\pi^2} \int_I \mathrm{d}z\, \frac{\mathrm{e}^{-\mathrm{i}z}}{z^N(z^2 - \alpha)} \\
&= \frac{-2\mathrm{i}\pi}{(N-1)!} \frac{\mathrm{d}^{N-1}}{\mathrm{d}\alpha^{N-1}}\bigg|_{\alpha=4\pi^2} [R_+(\alpha) + R_-(\alpha) + R_0(\alpha)]
\end{aligned} \tag{95}$$

where $R_\pm(\alpha)$ and $R_0(\alpha)$ are the residues of the last integrand at $z = \pm\sqrt{\alpha}$ and 0, respectively. They are readily computed, assuming first that $N = 2M + 1$ is odd:

$$\begin{aligned} R_+(\alpha) + R_-(\alpha) &= \frac{\cos\sqrt{\alpha}}{\alpha^{M+1}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\alpha^{r-M-1} \\ R_0(\alpha) &= -\sum_{r=0}^{M} \frac{(-1)^r}{(2r)!}\alpha^{r-M-1}. \end{aligned} \tag{96}$$

Hence

$$R_+(\alpha) + R_-(\alpha) + R_0(\alpha) = (-1)^{M+1}\sum_{s=0}^{\infty} \frac{(-1)^s}{(2M+2s+2)!}\alpha^s. \tag{97}$$

Using equations (94), (95) and (97) we then obtain

$$\begin{aligned} C_N &= \left(\frac{2\pi^2}{\mathrm{i}}\right)^N (-1)^{M+1}(-\mathrm{i})\frac{\mathrm{d}^{N-1}}{\mathrm{d}\alpha^{N-1}}\bigg|_{\alpha=4\pi^2} \sum_{s=0}^{\infty} \frac{(-1)^s}{(2M+2s+2)!}\alpha^s \\ &= (2\pi^2)^N \sum_{n=0}^{\infty} \frac{(N+n-1)!}{n!(3N+2n-1)!}(-4\pi^2)^n. \end{aligned} \tag{98}$$

The result is exactly the same for even N. It suffices now to observe that the last series alternates in sign and is decreasing to deduce

$$C_N = (2\pi^2)^N \frac{(N-1)!}{(3N-1)!}\left[1 + \mathrm{O}\left(\frac{1}{N}\right)\right]. \tag{99}$$

Our procedure for evaluating A_N is quite similar, and below we give only the main steps. From equations (86) and (91) we obtain

$$\begin{aligned} A_N &= (N-1)!\left(\frac{2\pi^2}{\mathrm{i}}\right)^N \frac{\mathrm{i}}{2\pi}\int_I \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}z} \frac{(5+2\eta_N)z^2 - 4\pi^2}{Z^{N+1}(z^2-4\pi^2)^{N+1}} \\ &= \frac{1}{N}\left(\frac{2\pi^2}{\mathrm{i}}\right)^N \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}^N}{\mathrm{d}\alpha^N}\bigg|_{\alpha=4\pi^2} \int_I \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}z}\left[\frac{5+2\eta_N}{z^{N-1}(z^2-\alpha)} - \frac{4\pi^2}{z^{N+1}(z^2-\alpha)}\right] \end{aligned} \tag{100}$$

and, after computing the residues at $z = \pm\sqrt{\alpha}$ and $z = 0$, we obtain

$$\begin{aligned} A_N &= \frac{(-2\pi^2)^N}{N}\frac{\mathrm{d}^N}{\mathrm{d}\alpha^N}\bigg|_{\alpha=4\pi^2} \sum_{s=0}^{\infty}\left[\frac{5+2\eta_N}{(N+2s)!} - \frac{4\pi^2}{(N+2s+2)!}\right](-\alpha)^s \\ &= \frac{(2\pi^2)^N}{N}\sum_{n=0}^{\infty}\frac{(N+n)!}{n!}\left[\frac{5+2\eta_N}{(3N+2n)!} - \frac{4\pi^2}{(3N+2n+2)!}\right](-4\pi^2)^n. \end{aligned} \tag{101}$$

Again, the last series alternates in sign and decreases, which entails

$$A_N = (5+2\eta_N)(2\pi^2)^N \frac{(N-1)!}{(3N)!}\left[1 + \mathrm{O}\left(\frac{1}{N}\right)\right]. \tag{102}$$

Finally, using equations (81), (99), (102) and (88) we obtain

$$\frac{\rho_N^{(0)}}{N} = \frac{5+2\eta_N}{3N}\left[1 + \mathrm{O}\left(\frac{1}{N}\right)\right] = \begin{cases} \dfrac{2}{3}\left[1 + \mathrm{O}\left(\dfrac{1}{N}\right)\right] & \text{for bosons} \\ \dfrac{1}{N}\left[1 + \mathrm{O}\left(\dfrac{1}{N}\right)\right] & \text{for fermions.} \end{cases} \tag{103}$$

The same procedure applies for all integer values of β, although the algebra becomes quite involved. The general result for bosons (and for any integer β) is

$$\lim_{N\to\infty} \frac{\rho_N^{(0)}}{N} = \frac{(\beta!)^4[(3\beta+1)!]^2}{[(2\beta)!]^2[(2\beta+1)!]^3}. \tag{104}$$

Our method does not adapt straightforwardly to the case of non-integer values of β, but there is clearly no reason to expect a different outcome for such intermediate values. Therefore, the Onsager–Penrose criterion (80) is *not* met for bosons, and we reach the conclusion that Bose–Einstein condensation is *possible* in the bosonic version of the N-body model discussed in section 3.

In the fermionic version, the result (103) is not conclusive, as explained after equation (80). It only points (not too surprisingly) to the absence of a quantum phase in the system.

6. The B_N model in one dimension

Subsequent to the seminal work of Calogero and Sutherland for the A_{N-1} system, the entire bound state spectrum of the Calogero model was obtained for the BC_N, D_N root systems [3, 29]. It is then natural to enquire whether in our case, can one at least obtain the exact ground state and radial excitation spectrum in the BC_N or D_N case? We now show that the answer to this question is in the affirmative.

Consider the BC_N Hamiltonian [29],

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + V\left(\sum_{i=1}^{N} x_i^2\right) + g\sum_{i=1}^{N-1}\left[\frac{1}{(x_i - x_{i+1})^2} + \frac{1}{(x_i + x_{i+1})^2}\right]$$
$$-G\sum_{i=2}^{N-1}\left[\left(\frac{1}{x_{i-1}-x_i} - \frac{1}{x_{i-1}+x_i}\right)\left(\frac{1}{x_i - x_{i+1}} + \frac{1}{x_i + x_{i+1}}\right)\right] + g_1\sum_{i=1}^{N}\frac{1}{x_i^2} \tag{105}$$

of which B_N, C_N and D_N are the special cases. We again restrict our attention to the sector of configuration space corresponding to a definite ordering of the particles as given by equation (4).

We start with the ansatz

$$\psi = P_{2k}(x)\phi(r)\left(\prod_{i=1}^{N}(x_i^2)^{\gamma/2}\right)\prod_{i=1}^{N-1}(x_i^2 - x_{i+1}^2)^{\beta} \tag{106}$$

where $r^2 = \sum_{i=1}^{N} x_i^2$. On substituting it in the Schrödinger equation for the B_N-Hamiltonian (105) we find that ϕ satisfies

$$\Phi''(r) + [N + 4k - 1 + 2N\gamma + 4(N-1)\beta]\frac{1}{r}\Phi'(r) + 2\,[E - V(r)]\,\Phi(r) = 0 \tag{107}$$

provided g and G are again related to β by equation (2), while g_1 is related to γ by

$$g_1 = \tfrac{1}{2}\gamma(\gamma - 1). \tag{108}$$

Here, $P_{2k}(x)$ is a homogeneous polynomial of degree $2k$ ($k = 0, 1, 2, \ldots$) in the particle coordinates and satisfies the generalized Laplace equation

$$\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\gamma\sum_{i=1}^{N}\frac{1}{x_i}\frac{\partial}{\partial x_i} + 4\beta\sum_{i=1}^{N-1}\frac{1}{(x_i^2 - x_{i+1}^2)}\left(x_i\frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}}\right)\right]P_{2k}(x) = 0. \tag{109}$$

Let us now specialize to the case of the oscillator potential, i.e. $V(r) = \frac{1}{2}\omega^2 r^2$. In this case, (107) is the well known radial equation for the oscillator problem in more than one dimension and its solution is

$$\Phi(r) = \exp(-\omega r^2/2) L_n^a(\omega r^2) \qquad n = 0, 1, 2, \ldots \tag{110}$$

where $L_n^a(x)$ is the associated Laguerre polynomial, while the energy eigenvalues are given by

$$E_n = \left[2n + 2k + \tfrac{1}{2}N + N\gamma + 2(N-1)\beta\right]\omega \tag{111}$$

with $a = E/\omega - 2n - 1$. The exact ground state is obtained from here when $n = k = 0$. The fact that $n = k = 0$ gives the exact ground-state energy of the system can be easily shown *à la* the A_{N-1} case by the method of supersymmetric quantum mechanics. It may be noted that for large N, the energy E is proportional to N so that like the A_{N-1} case, the B_N model also has a good thermodynamic limit. In contrast, note that the long-ranged B_N Calogero model does not have a good thermodynamic limit.

Are there homogeneous polynomial solutions of equation (109) of degree $2k$ ($k \geqslant 1$)? While we are unable to answer this question for any k, at least for small values of k ($k > 0$) there does not seem to be any solution to equation (109). For example, we have failed to find any polynomial solution of degrees two, four and six. Thus it appears that unlike the A_{N-1} case, in the BC_N case one is only able to obtain the ground-state and radial excitations over it.

Proceeding in the same way, the energy eigenvalues and eigenfunctions in the case of the Coulomb-like potential (34) are

$$E = -\frac{\alpha^2}{2\left[n + 2k + \frac{1}{2}(N-1) + N\gamma + 2(N-1)\beta\right]^2} \tag{112}$$

$$\Phi = \mathrm{e}^{-\sqrt{2|E|}r} L_n^b(2\sqrt{2|E|}r) \tag{113}$$

where $b = N - 2 + 4k + 2N\gamma + 4(N-1)\beta$. Again, so far we have been able to obtain solutions only in the case $k = 0$.

As in section 2, in the BC_N Hamiltonian (105), all of the particles are not being treated on the same footing. Again, one possibility is to add extra terms. Consider, for example,

$$H_1 = H + H' \tag{114}$$

where H is as given by equation (105), while H' has the form

$$H' = g\left[\frac{1}{(x_N - x_1)^2} + \frac{1}{(x_N + x_1)^2}\right] - G\left[\left(\frac{1}{x_N - x_1} - \frac{1}{x_N + x_1}\right)\left(\frac{1}{x_1 - x_2} + \frac{1}{x_1 + x_2}\right) + \left(\frac{1}{x_{N-1} - x_N} - \frac{1}{x_{N-1} + x_N}\right)\left(\frac{1}{x_N - x_1} + \frac{1}{x_N + x_1}\right)\right]. \tag{115}$$

One can now run through the arguments as given above and show that the eigenstates for both the oscillator and Coulomb-like potentials have the same form as given above except that in the term multiplying β, $N-1$ is replaced by N at all places including in the Laplace equation (109). However, now we find that there are indeed solutions to the Laplace equation (109) (with $N-1$ replaced by N). In particular, the solution for any N ($\geqslant 4$) and $k = 4$ is given by

$$P_{k=4}(x) = a\sum_{i=1}^{N} x_i^4 + b\sum_{i<j}^{N} x_i^2 x_j^2 \tag{116}$$

where

$$\frac{b}{a} = -2\left[\frac{3 + 8\beta + 2\gamma}{N - 1 + 2(N-1)\gamma + 4(N-2)\beta}\right]. \tag{117}$$

As in the A_{N-1} case, we again find that even though the Laplace equation (109) is only invariant under cyclic permutations, the solution is, in fact, invariant under the permutation of any two coordinates. It will be interesting to try to find solutions for higher values of k and to study the full degeneracy of the spectrum.

Besides these two, one can obtain part of the spectra including the ground state for several other potentials but we shall not discuss them here.

7. BC_N model in one dimension with periodic boundary condition

Following the work of Sutherland [25] on the A_{N-1} root system, the exact ground state as well as the excitation spectrum were also obtained in the case of the BC_N, D_N root systems [3]. It is then worth enquiring whether, in our case, one can also obtain the ground state and the excitation spectrum. As a first step in that direction, we shall obtain the exact ground state of the BC_N model with a periodic boundary condition.

The Hamiltonian for the BC_N case is given by

$$\begin{aligned} H = & -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\frac{\pi^2}{L^2}\sum_{i=1}^{N}\left[\frac{1}{\sin^2\frac{\pi}{L}(x_i - x_{i+1})} + \frac{1}{\sin^2\frac{\pi}{L}(x_i + x_{i+1})}\right] \\ & + g_1\frac{\pi^2}{L^2}\sum_i\frac{1}{\sin^2\frac{\pi}{L}x_i} + g_2\frac{\pi^2}{L^2}\sum_i\frac{1}{\sin^2\frac{2\pi}{L}x_i} \\ & - G\frac{\pi^2}{L^2}\sum_{i=1}^{N}\left[\cot\frac{\pi}{L}(x_{i-1} - x_i) - \cot\frac{\pi}{L}(x_{i-1} + x_i)\right] \\ & \times\left[\cot\frac{\pi}{L}(x_i - x_{i+1}) + \cot\frac{\pi}{L}(x_i + x_{i+1})\right]. \end{aligned} \tag{118}$$

We again restrict our attention to the sector of the configuration space corresponding to a definite ordering of the particles as given by equation (2). For this case, we start with a trial wavefunction of the form

$$\Psi_0 = \prod_{i=1}^{N}\sin^\gamma\theta_i\prod_{i=1}^{N}(\sin^2 2\theta_i)^{\gamma_1/2}\prod_{i=1}^{N}[\sin^2(\theta_i - \theta_{i+1})]^{\beta/2}\prod_{i=1}^{N}[\sin^2(\theta_i + \theta_{i+1})]^{\beta 2} \tag{119}$$

($\theta_i = \pi x_i/L$) and substitute it in the Schrödinger equation for the Hamiltonian (118). We find that it is indeed a solution provided g and G are again related to β by equation (2), while g_1, g_2 are related to γ, γ_1 by

$$g_1 = \tfrac{1}{2}\gamma[\gamma + 2\gamma_1 - 1] \qquad g_2 = 2\gamma_1(\gamma_1 - 1). \tag{120}$$

The corresponding ground-state energy turns out to be

$$E_0 = \frac{N\pi^2}{2L^2}(\gamma + \gamma_1 + 2\beta)^2. \tag{121}$$

The fact that this is indeed the ground-state energy can be easily proved as in sections 2 and 3.

8. N-body problem in D dimensions

Having obtained some results for the N-body problem (1) in one dimension, we study generalization to higher dimensions. Let us consider the following model in D dimensions:

$$H = -\frac{1}{2}\sum_{i=1}^{N}\vec{\nabla}_i^2 + g\sum_{i=1}^{N-1}\frac{1}{(\vec{r}_i - \vec{r}_{i+1})^2} - G\sum_{i=2}^{N-1}\frac{(\vec{r}_{i-1} - \vec{r}_i).(\vec{r}_i - \vec{r}_{i+1})}{(\vec{r}_{i-1} - \vec{r}_i)^2(\vec{r}_i - \vec{r}_{i+1})^2} + V\left(\sum_{i=1}^{N}\vec{r}_i^2\right). \tag{122}$$

On using the ansatz,

$$\psi = \left(\prod_{i=1}^{N-1} \left| \vec{r}_i - \vec{r}_{1+1} \right|^{\beta} \right) \phi(r) \qquad r^2 = \sum_{i=1}^{N} \vec{r}_i^{\,2} \tag{123}$$

in the Schrödinger equation for the Hamiltonian (122), it can be shown that $\phi(r)$ satisfies

$$\phi''(r) + [DN - 1 + 2(N-1)\beta]\frac{1}{r}\phi'(r) + 2(E - V(r))\phi(r) = 0 \tag{124}$$

provided g and G are related to β by

$$g = \beta^2 + (D-2)\beta \qquad G = \beta^2. \tag{125}$$

Equation (124) is easily solved in the case of the oscillator potential (i.e. $V(r) = \frac{1}{2}\omega^2 r^2$) yielding the energy eigenstates as

$$\phi(r) = \exp\left(-\tfrac{1}{2}\omega r^2\right) L_n^b(\omega r^2) \tag{126}$$

$$E_n = \left[2n + (N-1)\beta + \tfrac{1}{2}DN\right]\omega. \tag{127}$$

Here $b = E/\omega - 2n - 1$. It may be noted that as in all other higher-dimensional many-body problems, one has only obtained a part of the energy eigenvalue spectrum which, however, includes the ground state. In particular, the ground-state energy eigenvalue and eigenfunction is given by

$$E_0 = \left[(N-1)\beta + \tfrac{1}{2}DN\right]\omega \tag{128}$$

$$\psi_0 = \exp\left(-\tfrac{1}{2}\omega \sum_{i=1}^{N} r_i^2\right) \prod_{i=1}^{N-1} \left|\vec{r}_i - \vec{r}_{i+1}\right|^{\beta}. \tag{129}$$

As expected, for $D = 1$ these results go over to those obtained in section 2. The fact that this is indeed the ground-state energy can be easily proved by again using a supersymmetric formulation [21].

At this point it is worth asking whether the probability distribution for N particles (at least for some D (> 1)) can be mapped to some known random matrix ensemble? In this context we recall that in the case of the Calogero-type model, it has been shown that in two space dimensions $|\psi_0|^2$ can be mapped to a complex random matrix [30]. Using this identification one was able to calculate all the correlation functions of the corresponding many-body theory and show the absence of long-range order, but the presence of an off-diagonal long-range order in that theory. Unfortunately, so far as we are aware of, the answer to this question is unknown in this particular case. We hope that at least in the case of two space dimensions, where $|\psi_0|^2$ for our model is given by

$$|\psi_0(z_i)|^2 = C \exp\left(-\omega \sum_{i=1}^{N} |z_i|^2\right) \prod_{i=1}^{N-1} |z_i - z_{i+1}|^{2\beta} \tag{130}$$

$|\psi_0|^2$ can be mapped to some variant of the short-range Dyson model.

Finally, we observe that the ground state and a class of excited states can also be obtained in D dimensions in the case where the oscillator potential is replaced by the N-body Coulomb-like potential $V(r) = -\alpha/\sqrt{\sum r_i^2}$, because the resulting equation (124) is essentially the radial equation for the Coulomb potential. In particular, the energy eigenvalues and eigenfunctions are given by

$$E_n = -\frac{\alpha^2}{2\left[n + \frac{1}{2}(DN - 1) + (N - 1)\beta\right]^2} \tag{131}$$

$$\psi_n = \exp(-\sqrt{2|E|}r) L_n^{b'}(2\sqrt{2|E|r}) \left(\prod_{i=1}^{N-1} |\boldsymbol{r}_i - \boldsymbol{r}_{i+1}|^\beta \right) \tag{132}$$

where $b' = DN - 2 + 2(N - 1)\beta$. It may again be noted that, whereas the ground-state energy is linear in β in the oscillator case, it is not so in the case of the Coulomb-like N-body potential.

9. Short-range model in two dimensions with novel correlations

A few years back, Murthy *et al* [31] considered a model in two dimensions with two- and three-body long-ranged interactions and obtained the exact ground state and a class of excited states. The interesting feature of this model was that all these states had a built-in novel correlation of the form $|X_{ij}|^g$ where

$$X_{ij} = x_i y_j - x_j y_i. \tag{133}$$

It is then natural to enquire whether one can construct a model in two dimensions and obtain ground and few excited states of the system all of which would have a built-in short-range correlation of the form

$$X_{j,j+1} = x_j y_{j+1} - y_j x_{j+1}. \tag{134}$$

We now show that this is indeed possible. Let us consider the following Hamiltonian:

$$H = -\frac{1}{2}\sum_{i=1}^{N} \vec{\nabla}_i^2 + \frac{\omega^2}{2}\sum_{i=1}^{N} \vec{r}_i^{\,2} + g\sum_{i=1}^{N-1} \frac{\vec{r}_i^{\,2} + \vec{r}_{i+1}^{\,2}}{X_{i,i+1}^2} - G\sum_{i=2}^{N-1} \frac{\vec{r}_{i-1}\cdot\vec{r}_{i+1}}{X_{i-1,i}X_{i,i+1}} \tag{135}$$

where $X_{i,i+1}$ is as given by equation (134). We start with the ansatz

$$\psi(x_i, y_i) = \left[\prod_{i=1}^{N-1} X_{i,i+1}^\beta\right] \exp\left(-\tfrac{1}{2}\omega \sum_i \vec{r}_i^{\,2}\right)\phi(x_i, y_i). \tag{136}$$

On substituting the ansatz in the Schrödinger equation $H\psi = E\psi$, one finds that ϕ satisfies the equation

$$\left[-\tfrac{1}{2}\sum_{i=1}^{N} \vec{\nabla}_i^2 + \omega\sum_{i=1}^{N} \vec{r}_i \dot{\vec{\nabla}}_i + \beta\sum_{i=1}^{N-1} \frac{1}{X_{i,i+1}}\left(x_{i+1}\frac{\partial}{\partial y_i} - y_{i+1}\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial x_{i+1}} - x_i\frac{\partial}{\partial y_{i+1}}\right)\right]\phi$$
$$= \left(E - [N + 2(N - 1)\beta]\omega\right)\phi \tag{137}$$

provided g and G are related by (2). It is interesting to note that even though we are considering a novel correlation model in two dimensions, the relationship between g and G is as in the case of our one-dimensional model. We do not know whether this has any deep significance.

We conclude from here that ψ, as given by equation (136), with ϕ a constant, is the ground state of the system with the corresponding ground-state energy being

$$E_0 = [N + 2(N - 1)\beta]\omega. \tag{138}$$

Let us note that, like the relationship between coupling constants, the ground-state energy also has essentially the same form as that of the one-dimensional short-range A_{N-1} model as given by equation (13). That one has indeed obtained the ground state can be proved as before.

As in other many-body problems in two and higher dimensions, we are unable to find the complete excited-state spectrum. However, a class of excited states can be obtained from (137). To that end we introduce the complex coordinates

$$z = x + \mathrm{i}y \qquad z^* = x - \mathrm{i}y \qquad \partial \equiv \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)$$
$$\partial^* \equiv \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right). \tag{139}$$

In terms of these coordinates, the differential equation (137) takes the form

$$\left[-2\sum_{i=1}^{N} \partial_i \partial_i^* + 2\beta \sum_{i=1}^{N-1} \frac{\left(z_{i+1}\partial_i - z_i\partial_{i+1} + z_i^*\partial_{i+1}^* - z_{i+1}^*\partial_i^*\right)}{\left(z_i z_{i+1}^* - z_i^* z_{i+1}\right)} \right.$$
$$\left. +\omega \sum_{i=1}^{N} \left(z_i\partial_i + z_i^*\partial_i^*\right) - (E - E_0)\right]\phi = 0. \tag{140}$$

Now it is readily proved shown that the Hamiltonian H commutes with the total angular momentum operator $L = \sum_{i=1}^{N}(z_i\partial_i - z_i^*\partial_i^*)$, so that one can classify solutions according to their angular momentum: $L\phi = l\phi$.

On defining $t = \omega\sum_i z_i z_i^*$ and letting $\phi \equiv \phi(t)$ it is easily shown that $\phi(t)$ satisfies

$$t\phi''(t) + \left[\frac{E_0}{\omega} - t\right]\phi'(t) + \left(\frac{E - E_0}{2\omega}\right)\phi(t) = 0 \tag{141}$$

where E_0 is as given by equation (138). Hence the allowed solutions with $l = 0$ are

$$E = E_0 + 2n\omega \qquad \phi(t) = L_n^{E_0/\omega - 1}(t). \tag{142}$$

Solutions with angular momentum $l > 0$ or $l < 0$ can similarly be obtained by introducing $t_z = \omega\sum_i z_i^2$ or $t_{z^*} = \omega\sum_i (z_i^*)^2$. For example, let $\phi = \phi(t_z)$. Then equation (140) reduces to

$$2\omega t_z \frac{\mathrm{d}\phi}{\mathrm{d}t_z} = (E - E_0)\phi. \tag{143}$$

This is the well known Euler equation whose solutions are just monomials in t_z. The solution is given by $\phi(t_z) = t_z^m (m > 0)$, and hence the angular momentum $l = 2m$, while the energy eigenvalues are $E = E_0 + 2m\omega = E_0 + l\omega$. Furthermore, we can combine these solutions with the $l = 0$ solutions obtained above and obtain a tower of excited states. For example, let us define $\phi(z_i, z_i^*) = \phi_1(t)\phi_2(t_z)$, where ϕ_1 is a solution with $l = 0$, while ϕ_2 is the solution with $l > 0$. On using $\phi_2(t_z) = t_z^m$ it is easily shown that ϕ_1 again satisfies a confluent hypergeometric equation,

$$t\phi_1''(t) + \left[\frac{E_0}{\omega} + 2m - t\right]\phi_1'(t) + \left(\frac{E - E_0}{2\omega} + m\right)\phi_1(t) = 0. \tag{144}$$

Hence the energy eigenvalues are given by $E - E_0 = (2n_r + 2m)\omega$. One may repeat the procedure to obtain exact solutions for a tower of states with $l < 0$.

10. Summary

In this paper we have discussed an N-body problem in one dimension and presented its exact ground state on a circle and most likely the entire spectrum on a real line. There are several similarities as well as differences between the model discussed here and Calogero–Sutherland-type models and it might be worthwhile to compare the salient features of the two.

(a) Whereas in CSM the interaction is between all neighbours, in our case the interaction is only between nearest- and next-to-nearest-neighbours. Note, however, that in both the cases it is an inverse square interaction.

(b) Whereas in CSM (in one dimension) there is only a two-body interaction, both two- and three-body interactions are required in our model for partial (or possibly exact) solvability on a real line.

(c) Whereas the complete bound state spectrum is obtained in the Sutherland model (periodic potential) or if there is an external harmonic or Coulomb-like N-body potential as given by equation (34) and in the case of both A_{N-1} and BC_N root systems, it is not clear whether this is so in our case even though it is likely that this may be so in the A_{N-1} case.

(d) Whereas our system, both on a line and on a circle, has a good thermodynamic limit (i.e. E/N is finite for large N), CSM does not have a good thermodynamic limit in either case and E/N diverges like N for large N.

(e) In both the cases, the norm of the ground-state wavefunction can be mapped to the joint probability density function of the eigenvalues of some random matrix. Using this correspondence, in both cases, one is able to calculate one- and two-point functions. However, whereas in the CSM this is possible only at three values of the coupling (corresponding to orthogonal, unitary or simplistic random matrices), in our case the correlation functions can be computed analytically for any integral or half-integral values of the coupling, while numerically it can be done for any positive β.

(f) In the CSM case with an external potential of the form

$$V\left(\sum_i x_i^2\right) = A\sum_{i=1}^{N} x_i^2 + B\left(\sum_i x_i^2\right)^2 + C\left(\sum_i x_i^2\right)^3 \tag{145}$$

it has been shown [24] that the norm of the ground-state wavefunction can be mapped to a random matrix corresponding to branched polymers. It is not known whether a similar mapping is possible in our case.

(g) A multi-species generalization of CSM has been done [32], it is not clear whether a similar generalization is possible in our case or not.

(h) Generalization to D dimensions ($D > 1$) is possible in CSM as well as in our model and in both cases one is able to obtain only a partial spectrum including the ground state. In both cases, both two- and three-body interactions are required. Whereas our system has a good thermodynamic limit in any dimension D, the CSM does not have a good thermodynamic limit in any dimension. However, whereas the norm of the ground-state wavefunction can be mapped to complex random matrices in the CSM case in two dimensions [30], no such mapping has so far been possible in our case for $D > 1$.

(i) A model with novel correlations is possible in two dimensions in both the cases [31] but unlike CSM, our system has a good thermodynamic limit.

(j) In the CSM, it has been possible to obtain the entire spectrum algebraically by using supersymmetry and shape invariance [33]. It would be nice if a similar thing could also be done in our model. Furthermore, in the CSM, one has also written down the supersymmetric version of the model [34]. It would be worth enquiring whether a similar thing could also be done in our model.

(k) In the CSM-type models, one knows the various exactly solvable problems in which the N particles interact pairwise by two-body interaction [35]. The question one would like to ask in our context is: what are the various exactly solvable problems in one dimension in which the N particles have only nearest- and next-to-nearest-neighbour interactions?

(l) In the CSM, not only one- and two-point but even n-point correlation functions are known. It would be nice if the same were also possible in the present context.

(m) *À la* Haldane–Shastry spin models [36], can we also construct spin models in the context of our model?

(n) Unlike CSM, in our case the off-diagonal long-range order is non-zero in the bosonic version of the many-body theory in one dimension. Note, however, that the off-diagonal long-range order is non-zero in the CSM in two dimensions.

Acknowledgments

SRJ acknowledges the invitation by Professor S N Behera and the hospitality of the Institute of Physics, Bhubaneswar where this work was initiated, while AK would like to thank the members of the Laboratoire de Physique Mathématique of Montpellier University for warm hospitality during his trip there as a part of the Indo–French Collaboration Project 1501-1502. The work of GA and AK is supported in part by the Indo-French Centre for promotion of advanced Research (CEFIPRA Project 1501-1502).

Appendix

Proof of the representation (85) of C_N

By construction, the square of the wavefunction (70) is a symmetrical function of all its arguments, so that we can write equation (75) as well:

$$C_N = N! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{N-1}} \mathrm{d}x_N \, |\psi_N(x_1, \ldots, x_N)|^2 \tag{A1}$$

where the particle coordinates are now properly ordered. We are thus allowed to substitute ϕ_N for ψ_N in (A1) and obtain from equations (71) and (83)

$$C_N = N! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{N-1}} \mathrm{d}x_N \prod_{n=1}^{N} S(x_n - x_{n+1})^2. \tag{A2}$$

Changing the integration variables $(x_1, x_2, \ldots, x_N)$ to $(\triangle_1, \triangle_2, \ldots, \triangle_{N-1}, x_N)$, where

$$\triangle_n = x_n - x_{n+1} \qquad (n = 1, \ldots, N-1) \tag{A3}$$

one easily obtains

$$C_N = N! \int_0^1 \mathrm{d}\triangle_1 \int_0^{1-\triangle_1} \mathrm{d}\triangle_2 \cdots \int_0^{1-\triangle_1-\cdots-\triangle_{N-2}} \mathrm{d}\triangle_{N-1}$$
$$\times \int_0^{1-\sum_{p=1}^{N-1}\triangle_p} \mathrm{d}x_N \prod_{n=1}^{N-1} S(\triangle_n)^2 S(x_N - x_1)^2. \tag{A4}$$

Since $x_N - x_1 = -\sum_{p=1}^{N-1} \triangle_p$ is, in fact, independent of x_N in the new set of variables, equation (A4) becomes, also using $S(-x) = S(1-x)$:

$$C_N = N! \int_0^1 \mathrm{d}\triangle_1 \int_0^{1-\triangle_1} \mathrm{d}\triangle_2 \cdots \int_0^{1-\triangle_1-\cdots-\triangle_{N-2}} \mathrm{d}\triangle_{N-1} \times \left(1 - \sum_{p=1}^{N-1} \triangle_p\right) \prod_{n=1}^{N-1} S(\triangle_n)^2 S\left(1 - \sum_{p=1}^{N-1} \triangle_p\right)^2. \tag{A5}$$

It is now convenient to introduce the extra variable

$$\triangle_N = 1 - \sum_{p=1}^{N-1} \triangle_p \tag{A6}$$

and to recast equation (A5) in the form

$$\begin{aligned} C_N &= N! \int_0^1 \mathrm{d}\triangle_1 \int_0^1 \mathrm{d}\triangle_2 \cdots \int_0^1 \mathrm{d}\triangle_{N-1} \int_0^1 \mathrm{d}\triangle_N \, \delta\left(1 - \sum_{p=1}^{N} \triangle_p\right) \triangle_N \prod_{n=1}^{N} S(\triangle_N)^2 \\ &= N! \int_0^1 \mathrm{d}\triangle_1 \cdots \int_0^1 \mathrm{d}\triangle_N \, \delta\left(1 - \sum_{p=1}^{N} \triangle_p\right) \frac{1}{N} \sum_{m=1}^{N} \triangle_m \prod_{n=1}^{N} S(\triangle_n)^2 \\ &= (N-1)! \int_0^1 \mathrm{d}\triangle_1 \cdots \int_0^1 \mathrm{d}\triangle_N \, \delta\left(1 - \sum_{p=1}^{N} \triangle_p\right) \prod_{n=1}^{N} S(\triangle_n)^2. \end{aligned} \tag{A7}$$

In the second equality, we have used the fact that, apart from the factor $\triangle_N$, the integrand and the integration range are completely symmetrical in the variables $(\triangle_1, \ldots, \triangle_N)$. Finally, the integration over these variables factorizes after introducing the representation

$$\delta\left(1 - \sum_{p=1}^{N} \triangle_p\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x\left(1 - \sum_{p=1}^{N} \triangle_p\right)} \tag{A8}$$

and interchanging the x- and $\triangle$-integrations. This produces equation (85).

Proof of the representation (86) of A_N

Proceeding along the same lines, we first put the expression (82) of A_N in the form

$$A_N = (N-1)! \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{N-2}} \mathrm{d}x_{N-1} \, \phi_N(x_1, \ldots, x_{N-1}, 0) R_N(x_1, \ldots, x_{N-1}) \tag{A9}$$

where

$$\begin{aligned} R_N(x_1, \ldots, x_{N-1}) &= \int_0^{x_{N-1}} \mathrm{d}x \, \phi_N(x_1, \ldots, x_{N-1}, x) \\ &\quad \pm \int_{x_{N-1}}^{x_{N-2}} \mathrm{d}x \, \phi_N(x_1, \ldots, x, x_{N-1}) + \cdots + \int_{x_1}^1 \mathrm{d}x \, \phi_N(x, x_1, \ldots, x_{N-1}) \\ &= \int_0^{x_{N-1}} \mathrm{d}x \, \phi_N(x_1, \ldots, x_{N-1}, x) + \int_{x_1}^1 \mathrm{d}x \, \phi_N(x, x_1, \ldots, x_{N-1}) \\ &\quad + \sum_{p=1}^{N-2} \nu_p \int_{x_{p+1}}^{x_p} \mathrm{d}x \, \phi_N(x_1, \ldots, x_p, x, x_{p+1}, \ldots, x_N). \end{aligned} \tag{A10}$$

Here, $\nu_p = 1$ ($\nu_p = (-1)^p$) for bosons (fermions) and we have used the restriction to odd N in the second case. Thanks to the periodicity and the cyclic symmetry of ϕ_N, the first two terms in the last expression above can be collected to give

$$\int_{x_1-1}^{x_{N-1}} \mathrm{d}x\, \phi_N(x, x_1, \ldots, x_{N-1}).$$

Hence R_N becomes (with $x_N = x_1 - 1$)

$$\begin{aligned} R_N(x_1,\ldots,x_{N-1}) &= \sum_{p=1}^{N-1} \nu_p \int_{x_{p+1}}^{x_p} \mathrm{d}x\, \phi_N(x_1,\ldots,x_p,x,x_{p+1},\ldots,x_{N-1}) \\ &= \sum_{p=1}^{N-1} \nu_p \prod_{n=1}^{N-1} S(x_n - x_{n+1}) \int_{x_{p+1}}^{x_p} \mathrm{d}x\, S(x-x_{p+1})S(x_p - x) \qquad (n \neq p) \\ &= \sum_{p=1}^{N-1} \nu_p \sum_{n=1}^{N-1} S(x_n - x_{n+1}) S_2(x_p - x_{p+1}) \qquad (n \neq p) \end{aligned} \tag{A11}$$

according to the definition (84). We also have

$$\phi_N(x_1,\ldots,x_{N-1},0) = \prod_{m=1}^{N-2} S(x_m - x_{m+1}) S(x_{N-1}) S(x_1). \tag{A12}$$

Inserting equations (A11) and (A12) in equation (A9) and introducing as before the new integration variables $\triangle_n \equiv x_n - x_{n+1}$ ($n = 1,\ldots,N-2$) and x_{N-1}, we obtain

$$\begin{aligned} A_N = (N-1)! \int_0^1 \mathrm{d}\triangle_1 \int_0^{1-\triangle_1} \mathrm{d}\triangle_2 \cdots \int_0^{1-\triangle_1-\cdots-\triangle_{N-3}} \mathrm{d}\triangle_{N-2} \\ \times \int_0^{\triangle_{N-1}} \mathrm{d}x_{N-1} \prod_{m=1}^{N-2} S(\triangle_m) S(x_{N-1}) S(x_{N-1} - \triangle_{N-1}) \\ \times \sum_{p=1}^{N-1} \nu_p \prod_{n=1}^{N-1} S(\triangle_n) S_2(\triangle_p) \qquad (n \neq p) \end{aligned} \tag{A13}$$

where $\triangle_{N-1} = 1 - \sum_{p=1}^{N-2} \triangle_p$. The integration over x_{N-1} gives the factor $S_2(\triangle_{N-1})$ in place of $S(x_{N-1})S(x_{N-1} - \triangle_{N-1})$, so that

$$\begin{aligned} A_N = (N-1)! \int_0^1 \mathrm{d}\triangle_1 \cdots \int_0^1 \mathrm{d}\triangle_{N-1}\, \delta\left(1 - \sum_{p=1}^{N-1} \triangle_p\right) \\ \times \left[\prod_{m=1}^{N-2} S(\triangle_m) S_2(\triangle_{N-1}) \sum_{p=1}^{N-1} \nu_p \prod_{n=1}^{N-1} S(\triangle_n) S_2(\triangle_p) \right] \qquad (n \neq p). \end{aligned} \tag{A14}$$

On taking into account the complete symmetry of the integration measure, one finds that the square bracket in equation (A14) can be replaced by

$$[\cdots] = \prod_{m=1}^{N-2} S(\triangle_m)^2 S_2(\triangle_{N-1})^2 + \eta_N \prod_{m=1}^{N-3} S(\triangle_m)^2 [S(\triangle_{N-2}) S_2(\triangle_{N-2})][S(\triangle_{N-1}) S_2(\triangle_{N-1})] \tag{A15}$$

where η_N is as defined in equation (88). Finally, one obtains the factorization of the multiple integral in equation (A14) by again using the representation (A8) of the δ measure (with $(N-1)$ in place of N). This entails equation (86).

A final remark may be in order. Alternative, equivalent forms of the representations (85) and (86) would be obtained by relying on Fourier expansions instead of Fourier integrals, that is by considering the integrands in equations (A5) and (A13) not as functions with compact supports $[0, 1]^N \subset \mathcal{R}^N$, respectively $[0, 1]^{N-1} \subset \mathcal{R}^{N-1}$, but as periodic functions (this would amount to modifying equation (A8) accordingly). It turns out, however, that the resulting representations of C_N and A_N (as Fourier series) are much less convenient for the explicit or asymptotic evaluations of these quantities.

References

[1] Calogero F 1969 *J. Math. Phys.* **10** 2191, 2197
Calogero F 1971 *J. Math. Phys.* **12** 419
[2] Sutherland B 1971 *J. Math. Phys.* **12** 246, 251
[3] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 314
Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 6
[4] Simons B, Lee P and Altshuler B 1994 *Phys. Rev. Lett.* **72** 64
Simons B, Lee P and Altshuler B 1992 *Two Dimensional Quantum Gravity and Random Surfaces* ed D Gross, T Piran and S Weinberg (Singapore: World Scientific)
[5] Mattis D C (ed) 1995 *The Many-Body Problem: an Encyclopedia of Exactly Solved Models in One Dimension* (Singapore: World Scientific)
[6] Mehta M L 1991 *Random Matrices* (New York: Academic)
[7] Srednicki M 1994 *Phys. Rev.* E **50** 888
Jain S R and Alonso D 1997 *J. Phys. A: Math. Gen.* **30** 4993
[8] Jain S R and Pati A K 1998 *Phys. Rev. Lett.* **80** 650
Jain S R 1993 *Phys. Rev. Lett.* **70** 3553
[9] Bohigas O, Giannoni M-J and Schmit C 1984 *Phys. Rev. Lett.* **52** 1
[10] Dyson F 1962 *J. Math. Phys.* **3** 140
[11] Haake F 1991 *Quantum Signatures of Chaos* (Heidelberg: Springer)
[12] Zemlyakov A N and Katok A B 1976 *Math. Not.* **18** 760
Richens P J and Berry M V 1981 *Physica* D **2** 495
Jain S R and Parab H D 1992 *J. Phys. A: Math. Gen.* **25** 6669
Jain S R and Lawande S V 1995 *Proc. Ind. Natl Sci. Acad.* A **61** 275
[13] Parab H D and Jain S R 1996 *J. Phys. A: Math. Gen.* **29** 3903
[14] Date G, Jain S R and Murthy M V N 1995 *Phys. Rev.* E **51** 198
[15] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 189
[16] Grémaud B and Jain S R 1998 *J. Phys. A: Math. Gen.* **31** L637
[17] Pandey A Private communication
[18] Bogomolny E, Gerland U and Schmit C 1999 *Phys. Rev.* E **59** R1315
[19] Jain S R and Khare A 1999 *Phys. Lett.* A **262** 35
(Jain S R and Khare A 1999 *Preprint* cond-mat/9904121)
[20] Auberson G, Jain S R and Khare A 2000 *Phys. Lett.* A **267** 293
(Auberson G, Jain S R and Khare A 1999 *Preprint* cond-mat/9912445)
[21] Cooper F, Khare A and Sukhatme U P 1995 *Phys. Rep.* **251** 267
[22] Khare A 1996 *J. Phys. A: Math. Gen.* **29** L45
[23] Ghosh P K and Khare A 1999 *J. Phys. A: Math. Gen.* **32** 2129
(Ghosh P K and Khare A 1998 *Preprint* solv-int/9808005)
[24] Jatkar D P and Khare A 1996 *Mod. Phys. Lett.* A **11** 1357
[25] Sutherland B 1971 *Phys. Rev.* A **4** 2019
[26] Penrose O and Onsager L 1956 *Phys. Rev.* **104** 576
[27] Lenard A 1964 *J. Math. Phys.* **5** 930
[28] Yang C N 1962 *Rev. Mod. Phys.* **34** 694
[29] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
[30] Khare A and Ray K 1997 *Phys. Lett.* A **230** 139
[31] Murthy M V N, Bhaduri R K and Sen D 1996 *Phys. Rev. Lett.* **76** 4103
Bhaduri R K, Khare A, Law J, Murthy M V N and Sen D 1997 *J. Phys. A: Math. Gen.* **30** 2557

[32] Krivnov V Ya and Ovchinnikov A A 1982 *Teor. Mat. Fiz.* **50** 155
[33] Ghosh P K, Khare A and Sivakumar M 1998 *Phys. Rev.* A **58** 821
[34] Freedman D Z and Mende P F 1990 *Nucl. Phys.* B **344** 317
[35] Calogero F 1975 *Lett. Nuovo Cimento* **13**
[36] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
Shastry B S 1988 *Phys. Rev. Lett.* **60** 639